Last Time! $\text{Curl}(\vec{v}) = \nabla \times \vec{v}$ $\vec{v} = (P, Q, R)$ $\Big\{$ $\text{div}(\vec{v}) = \nabla \cdot \vec{v}$

Prop! ① $\text{Curl}(\nabla f) = \vec{0}$ and ② $\text{div}(\text{curl}(\vec{v})) = 0$

Note! ① Intuitively the divergence of a vector field calculates "how badly does the v.f. want to leave a bounded set"
② The curl is a measure of "how swirly" a v.f. wants to be

Recasting Green's Theorem

Let $\vec{v} = \langle P, Q, 0 \rangle$ have cts, partial derivatives on an open region R containing D, where D is a closed region with a piecewise smooth boundary curve

Then $\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$

$$\int_{\partial D} \vec{v} \cdot (y'(t)\,\vec{i} - x'(t)\,\vec{j}) \cdot \frac{1}{|\vec{r}(t)|} \cdot ds = \iint_D \text{div}(\vec{v}) \, \partial A$$

$$\text{curl}(\vec{v}) = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \langle -Q_z, P_z, Q_x - P_y \rangle$$

$$\therefore \text{curl}(\vec{v}) \cdot \vec{k} = Q_x - P_y = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$$

↑ Green's Theorem

$$\operatorname{div}(\vec{v}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, 0 \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$\therefore \iint_{\partial D} \operatorname{div}(\vec{v}) \cdot dA = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \quad \rightsquigarrow \vec{w} = \langle -Q, P, 0 \rangle$$

Greens Theorem $\longrightarrow$ $= \int_{\partial D} \vec{w} \cdot d\vec{r} = \int_a^b \left(-Q x'(t) + P y'(t) \right) dt$

$$= \int_a^b \langle P, Q \rangle \cdot \langle y', x' \rangle \, dt$$

$\langle x, y \rangle$ an arc-length

$$= \int_{\partial D} \vec{v} \cdot \left(y'(t)\vec{i} - x'(t)\vec{j} \right) \frac{1}{|r'(t)|} \, ds$$

Point! Greens Theorem can be recast using
① Curl
② Divergence

These two ways of recasting Green's Theorem lead to two seperate generalizations of Green's Theorem

① Stokes's Theorem ② Divergence Theorem

§16.6 Parametric Surfaces (Not on Exam 3)

Def: A parametric surface is a function
$\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
for some domain in $\mathbb{R}^2$

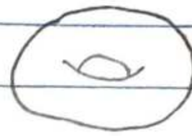
Idea: This is a "space curve in dimension 2"

Ex: A sphere of radius $r>0$ can be parameterized as: $\vec{S}(\theta,\varphi)=\langle r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)\rangle$
↑ from spherical coords.

on $D=[0,2\pi]\times[0,\pi]$

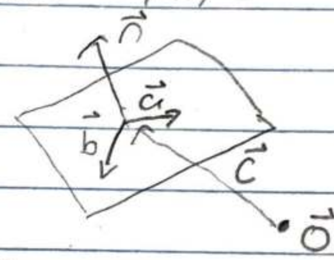
Ex: The torus has parameterization
$\vec{S}(u,v)=\langle (2+\sin(v))\cos(u), (2+\sin(v))\sin(u), \cos(v)\rangle$
On $D=[0,2\pi]\times[0,\pi]$

Torus →



Ex: every plane can be parametrized via

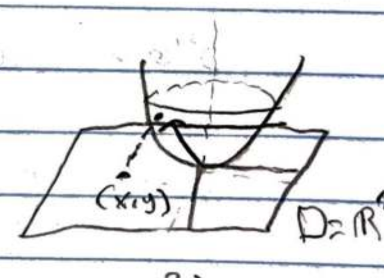
$\vec{S}(u,v)=u\vec{a}+v\vec{b}+\vec{c}$ for suitable $\vec{a},\vec{b},\vec{c}$
for $D=\mathbb{R}^2$



Idea: Π is just determined by points (u,v) in $\mathbb{R}^2$ via $\vec{a},\vec{b},\vec{c}$ and the above equation

Ex: Compute a parameterization for the paraboloid $z=x^2+2y^2$

Note: there are many ways to do this



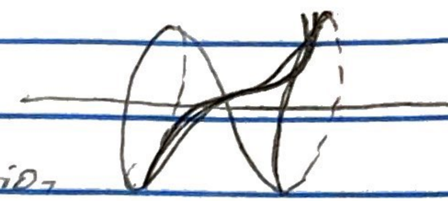
Sol①: $\vec{S}(x,y)=\langle x, y, x^2+2y^2\rangle$

Sol②: $\vec{S}(r,\theta)=\langle r\cos\theta, r\sin\theta, (r\cos\theta)^2+(2r\sin\theta)^2\rangle$
$D=[0,\infty]\times[0,2\pi]$ $=\langle r\cos\theta, r\sin\theta, r^2(1+\sin^2\theta)\rangle$

Sol③: $\vec{S}(r,\theta)=\langle\sqrt{2}\,r\cos\theta, r\sin\theta, 2r^2\rangle$

Ex Let $f(t)$ be a single variable function. The surface of revolution obtained by revolving f about the x-axis is parameterized by
$\vec{S}(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta) \rangle$

↳ Sub-ex: Let $f(x) = x^3$



the surface has parameterization
$\vec{s}(x,\theta) = \langle x, x^3\cos(\theta), x^3\sin(\theta) \rangle$